A LITTLE BIJECTION FOR AFFINE STANLEY SYMMETRIC FUNCTIONS

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Abstract. Little [13] developed a combinatorial algorithm to study the Schur-positivity of Stanley symmetric functions and the Lascoux-Schützenberger tree. We generalize this algorithm to affine Stanley symmetric functions, which were introduced recently in [7].

1. Introduction

A new family of symmetric functions, called *affine Stanley symmetric functions* were recently introduced in [7]. These symmetric functions $\tilde{F}_w$, indexed by affine permutations $w \in \tilde{S}_n$, are an affine analogue of the Stanley symmetric functions F_w which Stanley [16] introduced to enumerate the reduced decompositions of a permutation $w \in S_n$. Stanley symmetric functions were later shown to be stable limits of the *Schubert polynomials* $\mathfrak{S}_w$ [12, 1]. In the case that w is a *Grassmannian permutation* the Stanley symmetric function is equal to a Schur function. Shimozono conjectured and Lam [8] recently showed that the symmetric functions $\tilde{F}_w$ also had a geometric interpretation. When w is *affine Grassmannian* then $\tilde{F}_w$ represents a Schubert class in the cohomology $H^*(\mathcal{G}/\mathcal{P})$ of the affine Grassmannian and $\tilde{F}_w$ is called an *affine Schur function*. Affine Schur functions were introduced by Lapointe and Morse in [10], where they were called *dual k-Schur functions*.

A key property of the Stanley symmetric function F_w is that its expansion in terms of the Schur functions s_λ involves non-negative coefficients. This was proved by Edelman and Greene [2] using an insertion algorithm and separately by Lascoux and Schützenberger [12] via *transition formulae* for Schubert polynomials. These transition formulae lead to a combinatorial object known as the *Lascoux-Schützenberger tree* which allows one to write a Stanley symmetric function F_w in terms of other Stanley symmetric functions F_v labeled by permutations "closer to Grassmannian".

Answering a question of Garsia, Little [13] recently gave a combinatorial proof of the identity

$$\sum_{\substack{u=v\cdot t_{r,s}\\ l(u)=l(v)+1}} F_u(X) = \sum_{\substack{w=v\cdot t_{s',r}\\ l(w)=l(v)+1}} F_w(X),$$

from which the Lascoux-Schützenberger tree can be deduced. Here $v \in S_n$ is any permutation, $t_{r,s}$ denotes a transposition, and the summations are over $s > r$ and $s' < r$ respectively.

The aim of this article is to generalize Little's bijection and to prove an affine analogue of the above identity for the affine Stanley symmetric functions $\tilde{F}_w$. Our

Date: January 2006.
M.S. was supported in part by NSF DMS-0401012.

techniques are mainly combinatorial and the resulting bijection appears to be interesting in itself – for example, it seems to be closely related to the affine Chevalley formula. Unfortunately, we have been unable to use our affine Little bijection to prove that an arbitrary affine Stanley symmetric function expands positively in terms of affine Schur functions. This positivity follows from the results of Lam [8] combined with unpublished work of Peterson [15].

2. Affine Stanley symmetric functions

2.1. **Affine symmetric group.** Let $\tilde{S}_n$ be the affine symmetric group. It is a Coxeter group with simple reflections $\{s_i \mid i \in \mathbb{Z}/n\mathbb{Z}\}$ and relations $s_i^2 = 1$ for all i, $(s_i s_{i+1})^3 = 1$ for all i, and $(s_i s_j)^2 = 1$ for $i \neq j \pm 1$ not adjacent mod n.

One may realize $\tilde{S}_n$ as the set of all bijections $w : \mathbb{Z} \to \mathbb{Z}$ such that $w(i+n) = w(i) + n$ for all i and $\sum_{i=1}^n w(i) = \sum_{i=1}^n i$. In this realization, to specify an element it suffices to give the "window" $[w(1), w(2), \ldots, w(n)]$.

Given integers r and s such that $s \not\equiv r \mod n$, there is a unique element $t_{r,s} \in \tilde{S}_n$ such that, in the function notation, $t_{r,s}(r) = s$ and $t_{r,s}(s) = r$, and $t_{r,s}(i) = i$ for all i such that $i \not\equiv r \mod n$ and $i \not\equiv s \mod n$. We note that $t_{r,s} = t_{r',s'}$ if and only if there is an integer k such that $\{r, s\} = \{r' + kn, s' + kn\}$ as sets. In this notation $s_i = t_{i,i+1}$ for all i.

For $a_i \in \mathbb{Z}/n\mathbb{Z}$ we call $a = a_1 \cdots a_l$ a reduced word for $w \in \tilde{S}_n$ and write $a \in R(w)$ if $w = s_{a_1} \cdots s_{a_l}$ such that l is minimum. We call $l = \ell(w)$ the length of w. Let $v \lessdot w$ denote the covering relation of the *strong Bruhat order* $\leq$ on $\tilde{S}_n$. By definition $v \lessdot w$ if and only if there is a reflection $t_{r,s} \in \tilde{S}_n$ such that $w = v t_{r,s}$ and $\ell(w) = \ell(v) + 1$.

Say that $w \in \tilde{S}_n$ is a *right r-cover* of $v \in \tilde{S}_n$ if $v \lessdot w$ with $w = v t_{r,s}$ for $r < s$. Say that w is a *left r-cover* of v if $v \lessdot w$ with $w = v t_{s,r}$ where $s < r$. Let $\Psi_r^+(v)$ denote the set of r-right covers of v and $\Psi_r^-(v)$ denote the set of r-left covers of v.

Now let $S_n \subset \tilde{S}_n$ denote the symmetric group generated by $s_1, s_2, \ldots, s_{n-1}$. This is a parabolic subgroup of $\tilde{S}_n$. The minimal length coset representatives of $\tilde{S}_n/S_n$ are called *Grassmannian* and the set of such elements is denoted $\tilde{S}_n^-$.

2.2. **Cyclically decreasing permutations.** Let $a = a_1 a_2 \cdots a_l$ be a reduced word. Then a is called *cyclically decreasing* if

(1) The multiset $A = \{a_1, a_2, \ldots, a_l\}$ is a set.
(2) If $i, i+1 \in A$ then $i+1$ occurs before i in a, where indices are considered modulo n. In particular, if $n-1, 0 \in A$ then 0 appears before $n-1$ in a.

A permutation $w \in \tilde{S}_n$ is called *cyclically decreasing* if there is a reduced word for w which is cyclically decreasing. Say that a proper subset $I \subset \mathbb{Z}/n\mathbb{Z}$ is a *cyclic interval* if it has the form $I = \{i, i+1, i+2, \ldots, i+j\}$ with indices taken mod n. Let $(i+j) \cdots (i+1)i$ be the word of I. The cyclically decreasing elements of $\tilde{S}_n$ are characterized as follows.

Lemma 1. (1) *Let $A \subset \mathbb{Z}/n\mathbb{Z}$ be a proper subset. Then a word with underlying set A is cyclically decreasing if and only if it is a shuffle of the words of the maximal cyclic subintervals of A.*
(2) *Every cyclically decreasing word with underlying set A is a reduced word for the same element $w(A) \in \tilde{S}_n$.*
(3) *Every reduced word for $w(A)$ is cyclically decreasing with underlying set A.*

Proof. (1) holds by definition. Since reflections in different maximal cyclic subintervals commute, it follows that every cyclically decreasing word with underlying set A, is equivalent to a single canonical word, namely, the concatenation of the words of the maximal cyclic subintervals of A, with these words occurring in decreasing order by first element. This given word is a reduced word for some element $w(A) \in \tilde{S}_n$, so all the cyclically decreasing words with underlying set A are as well. This proves (2). Since there are no repeated reflections, the braid relations do not apply, and the only equivalences among reduced words for $w(A)$ are commutations between reflections in different maximal cyclic subintervals of A. Since the set of reduced words is connected by the Coxeter relations it follows that every reduced word for $w(A)$ is a shuffle of the prescribed sort. This proves (3). □

Lemma 1 has the following immediate consequence.

Corollary 2. *The strong Bruhat order on the set of cyclically decreasing elements in $\tilde{S}_n$ is isomorphic to the boolean lattice on proper subsets of $\mathbb{Z}/n\mathbb{Z}$.*

Now let $w \in \tilde{S}_n$ of length $l = \ell(w)$ and suppose $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ is a composition of l. An *α-decomposition* of w is an ordered r-tuple of cyclically decreasing affine permutations $(w^1, w^2, \ldots, w^r) \in \tilde{S}_n^r$ satisfying $\ell(w^i) = \alpha_i$ and $w = w^1 w^2 \cdots w^r$. The following definition is [7, Alternative Definition 2].

Definition 3. The affine Stanley symmetric function $\tilde{F}_w(X)$ is given by

$$\tilde{F}_w(X) = \sum_{\alpha} (\text{number of } \alpha\text{-decompositions of } w) \cdot x^\alpha$$

where the sum is over all compositions α of $\ell(w)$.

It is shown in [7] that $\tilde{F}_w(X)$ is always a symmetric function, though this fact will not be used in the current work. When $w \in S_n \subset \tilde{S}_n$ is a normal permutation then, the function $\tilde{F}_w$ is the usual Stanley symmetric function [16]. When $w \in S_n \cap \tilde{S}_n^-$ is a usual Grassmannian permutation, the function $\tilde{F}_w$ is equal to some Schur function s_λ.

When $w \in \tilde{S}_n^-$ is an affine Grassmannian permutation, then we say that $\tilde{F}_w(X)$ is an *affine Schur function* or *dual k-Schur function*. There is a bijection $\theta : w \leftrightarrow \lambda(w)$ between Grassmannian permutations $w \in \tilde{S}_n^-$ and partitions $\lambda(w)$ with no part greater than or equal to n. The bijection θ sends a permutation w with length l to a partition λ with l boxes. We may thus label the affine Schur functions by partitions $\tilde{F}_{\lambda(w)} := \tilde{F}_w$ so that $\deg(\tilde{F}_\lambda) = |\lambda|$. See [7] for details.

3. Affine Chevalley formula and affine Garsia-Little formula

3.1. **Affine Chevalley formula.** Let $\mathcal{G}/\mathcal{B}$ denote the affine flag variety of type A_{n-1}; see [6, 5]. The Bruhat decomposition of $\mathcal{G}$ induces a decomposition of $\mathcal{G}/\mathcal{B}$ into Schubert cells

$$\mathcal{G}/\mathcal{B} = \cup_{w \in \tilde{S}_n} \Omega_w.$$

Let $SS_w \in H^*(\mathcal{G}/\mathcal{B})$ denote the cohomology class dual to Ω_w.

The structure constants for the Schubert basis are denoted by

$$SS_u SS_v = \sum_{w \in \tilde{S}_n} c_{u,v}^w SS_w \tag{1}$$

for $u, v \in \tilde{S}_n$. The following is a translation of the general Chevalley rule (applicable to symmetrizable Kac-Moody groups) of Kostant and Kumar [5] for the special case of $\tilde{S}_n$.

Proposition 4. [5] *For $v, w \in \tilde{S}_n$ and any r, $c^w_{s_r,v}$ is zero unless $w \gtrdot v$. In this case, writing $w = vt_{a,b}$ with $a < b$, $c^w_{s_r,v}$ is the number of times that r occurs modulo n in the interval $[a, b-1]$.*

Conjecturally (1) holds with SS_w replaced by $\tilde{F}_w(X)$ everywhere. Note that the functions $\tilde{F}_w(X)$ are not linearly independent. In particular we have the following conjecture.

Conjecture 5.

$$\tilde{F}_{s_r}\tilde{F}_v = \sum_{w \gtrdot v} c^w_{s_r,v}\tilde{F}_w. \tag{2}$$

This conjecture follows from an affine Schensted algorithm developed in joint work with Lapointe and Morse [9]. It also follows from unpublished geometric work of Peterson [15] and the results in [8].

Example 6. Let $n = 4$. We use the window notation. Let $v = [2, 3, 0, 5]$ and $r = 2$. The elements w such that $c^w_{s_r,v}$ nonzero are $w_1 = [2, 5, 0, 3]$ and $w_2 = [2, 4, -1, 5]$. Now $w_1 = vt_{2,4}$ and 2 occurs once mod 4 in $\{2, 3\}$, and $w_2 = vt_{2,7}$ and 2 occurs twice mod 4 in $\{2, 3, 4, 5, 6\}$. Therefore $\tilde{F}_{s_2}\tilde{F}_v = \tilde{F}_{w_1} + 2\tilde{F}_{w_2}$.

3.2. **Affine Garsia-Little Formula.** The following, our main result, is an affine analogue of an identity for Stanley functions observed by Garsia [3], for which David Little [13] found a combinatorial proof.

Theorem 7. *For any $r \in \mathbb{Z}$ and $v \in \tilde{S}_n$,*

$$\sum_{u \in \Psi^-_r(v)} \tilde{F}_u = \sum_{w \in \Psi^+_r(v)} \tilde{F}_w. \tag{3}$$

Example 8. Let $n = 4$ and $v = [-1, 1, 4, 6] = s_3s_1s_0$. For $r = 1$ we have $\Psi^+_1(v) = \{[1, -1, 4, 6]\}$ and $\Psi^-_1(v) = \{[-3, 3, 4, 6], [-2, 1, 4, 7]\}$ and all the Chevalley coefficients are 1:

$$\tilde{F}_{[1,-1,4,6]} = \tilde{F}_{[-3,3,4,6]} + \tilde{F}_{[-2,1,4,7]}.$$

With the same n and v but $r = 2$, we have $\Psi^+_2(v) = \{[-1, 4, 1, 6], [-3, 3, 4, 6]\}$ and $\Psi^-_2(v) = \{[1, -1, 4, 6], [-1, 0, 5, 6]\}$ and all the Chevalley coefficients are 1:

$$\tilde{F}_{[-1,4,1,6]} + \tilde{F}_{[-3,3,4,6]} = \tilde{F}_{[1,-1,4,6]} + \tilde{F}_{[-1,0,5,6]}.$$

Using these equations together one may find the equation

$$\tilde{F}_{[-1,4,1,6]} = \tilde{F}_{[-2,1,4,7]} + \tilde{F}_{[-1,0,5,6]}$$

The latter two are the affine Schur functions indexed by the partitions $(2, 1, 1)$ and $(2, 2)$ respectively (see [7]).

Remark 9. Theorem 7 is a consequence of Conjecture 5. Note that $\tilde{F}_{s_r}$ is the Schur function s_1 for all r. Subtracting (2) for $\tilde{F}_{s_r}\tilde{F}_v$ and $\tilde{F}_{s_{r+1}}\tilde{F}_v$ one obtains (3).

3.3. **Garsia-Little Formula and the Lascoux-Schützenberger Tree.** Theorem 7 holds, with a slight modification in a special case (explained below), with all the affine objects (affine permutations, affine transpositions, affine Stanley symmetric functions) replaced by their usual S_n-counterparts. The resulting *Garsia-Little formula* implies that a Stanley symmetric function F_w is Schur-positive, as follows.

Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$ be a permutation. We set

$$\begin{aligned} r &= \max(i \mid \sigma_i > \sigma_{i+1}), \\ s &= \max(i > r \mid \sigma_i < \sigma_r), \\ I &= \{i < r \mid \sigma_i < \sigma_s \text{ and for all } j \in (i,r) \text{ we have } \sigma_j \notin (\sigma_i, \sigma_s)\}. \end{aligned}$$

Now let $\pi = \sigma \cdot t_{r,s}$. One can check that we have $\Psi_r^+(\pi) = \{\sigma\}$ and $\Psi_r^-(\pi) = \{\pi \cdot t_{i,r} \mid i \in I\}$. If $I \neq \emptyset$ then Theorem 7 reads

$$F_\sigma = \sum_{i \in I} F_{\pi \cdot t_{i,r}}.$$

The permutations $\pi \cdot t_{i,r}$ are the children of σ in the Lascoux-Schützenberger tree. When $I = \emptyset$, the corresponding equation fails to hold, but we declare σ to have a single child $1 \otimes \sigma = 1\,\sigma_1\sigma_2\cdots\sigma_n \in S_{n+1}$ and we note that $F_\sigma = F_{1\otimes\sigma}$. It can be shown that the process of (repeatedly) taking children eventually results in permutations which are Grassmannian, which are the leaves of the Lascoux-Schützenberger tree. Since the Stanley symmetric function indexed by σ is equal to the sum of those indexed by the children of σ we conclude that it is also the sum of those indexed by the leaves which are descendents of σ. A Stanley symmetric function indexed by a Grassmannian permutation is a Schur function, so in particular every Stanley symmetric function is Schur positive.

Unfortunately, a similar attempt to produce an "affine Lascoux-Schützenberger tree" fails because for an affine permutation $w \in \tilde{S}_n$ there maybe no permutation v and index r so that Theorem 7 involves only $\tilde{F}_w$ on one side. However, by solving simultaneous equations obtained from Theorem 7, we have so far always been able to express an affine Stanley symmetric function in terms of affine Schur functions, as demonstrated in Example 8. It is our hope that the methods of this paper will eventually lead to a combinatorial interpretation of the coefficients a_w^λ in the expansion $\tilde{F}_w = \sum_\lambda a_w^\lambda \tilde{F}_\lambda$ of affine Stanley symmetric functions in terms of affine Schur functions. These coefficients contain, for example, the 3-point, genus zero, Gromov Witten invariants of the Grassmannian, which are important numbers in combinatorics, geometry, and representation theory; see [10, 7]. The numbers a_w^λ also include as a special case the structure constants for the multiplication of the homology of the affine Grassmannian in the Schubert basis; see [8, 15].

4. Affine Little Bijection

A *v-marked word* is a pair (a, i) where $a = a_1 \cdots a_l$ is a word with letters in $\{0, 1, \ldots, n-1\}$ and $i \in [1, l]$ is the index of a distinguished letter in a such that $a_1 \cdots \hat{a}_i \cdots a_l \in R(v)$.

We now define the *affine Little graph*. It is a directed graph whose vertices are the v-marked words. Given a v-marked word (a, i) with $a = a_1 \cdots a_l$, there is a unique directed edge $(a, i) \to (a', j)$ leaving (a, i), where a' is the word obtained from a by replacing the letter a_i by $a_i - 1 \pmod n$ and the index j is equal to i if a'

(a,i)	$p(a,i)$	$q(a,i)$
$3410\underline{2}321042$	2	5
$341013210 4\underline{2}$	2	3
$34\underline{1}01321041$	2	1
$340\underline{0}1321041$	2	3
$340\underline{4}1321041$	-6	2

FIGURE 1. Computation of ϕ^v.

is reduced and is otherwise the unique index $j \neq i$ such that $a_1 \cdots \hat{a_j} \cdots a_l \in R(v)$, whose existence and uniqueness follows by Lemma 21 (see Section 6).

It is not hard to see that each vertex (a,i) has a unique incoming edge $(b,k) \to (a,i)$: if a is reduced then $k = i$, and otherwise, $k \neq i$ is the unique index such that $a_1 \cdots \hat{a_k} \cdots a_l \in R(v)$, and in either case, b is obtained from a by replacing a_k by $a_k + 1$.

Since there are finitely many v-marked words, the connected components of the affine Little graph are finite directed cycles. It is clear from the definition that none of these cycles is a loop, that is, it is never the case that $(a,i) \to (a,i)$.

Say that the v-marked word (a,i) is *reduced* if a is a reduced word for some $w \in \tilde{S}_n$. Given a reduced v-marked word (a,i), write $\phi^v(a,i) = (b,j)$ where (b,j) is the first reduced v-marked word following (a,i) on the cycle of the affine Little graph containing (a,i). The map ϕ^v defines a bijection from the set of reduced v-marked words to itself. We call an application of ϕ^v the *affine Little algorithm*.

Let $w \gtrdot v$, $w = vt_{r,s}$ and $a = a_1 \cdots a_l \in R(w)$. By Proposition 16 there exists a unique $i \in [1,l]$ so that $a_1 \cdots \hat{a_i} \cdots a_l \in R(v)$ and (a,i) is v-marked. Therefore the set of reduced v-marked words is in bijection with $\bigcup_{w \gtrdot v} R(w)$, and ϕ^v can be regarded as a bijection from this set to itself.

Example 10. Let $n = 5$, $3410321042 \in R(v)$, $r = 2$, and $b = 3410\underline{2}321042 \in \Psi_r^+(v)$ where the distinguished reflection (the j-th) is underlined. The computation of $\phi^v(b,j)$ is shown below. The indices $p(a,i)$ and $q(a,i)$ are those that appear in the proof of Theorem 11. $\phi^v(b,j)$ is given by the last row. In Figure 1 the edges of the affine Little graph for v go from each v-marked word to the one in the next row. We give some additional data used in the proof of Theorem 11 with $r = 2$. We note that for the last row, literally $p(a,i) = -1$ and $q(a,i) = 7$. However $-6 < 2$ and $t_{-1,7} = t_{-6,2}$. In other words, one should identify the pairs (p,q) and (p',q') if there is a k such that $p' = kn + p$ and $q' = kn + q$.

Theorem 11. *The map ϕ^v restricts to a bijection*

$$\phi_r^v : R(\Psi_r^+(v)) \longrightarrow R(\Psi_r^-(v)).$$

Proof. Due to the symmetry of left and right r-covers and the bijectivity of ϕ^v, it suffices to show that ϕ^v maps $\Psi_r^+(v)$ into $\Psi_r^-(v)$.

Given a v-marked word (a,i) with $a = a_1 \cdots a_l$, let x and y be the elements of $\tilde{S}_n$ with reduced words $a_1 \cdots a_{i-1}$ and $a_{i+1} \cdots a_l$ respectively. Let $w = s_{a_1} \cdots s_{a_l}$ and $t = a_i$. Then $w = xs_ty = xy(y^{-1}s_ty) = vt_{p,q}$ where $y(p) = t$ and $y(q) = t+1$. Note also that if $s_ty > y$ (which occurs if a is reduced), then $p < q$, and if $s_ty < y$ then $p > q$. We shall use the notation $p = p(a,i)$ and $q = q(a,i)$ to emphasize the dependence on (a,i).

Let $b \in R(\Psi_r^+(v))$ and (b,j) the corresponding v-marked word. Let $\phi^v(b,j) = (c,k)$ where $c \in R(u)$. It is enough to show that $q(c,k) = r$, for if so, then since c is reduced, $p(c,k) < q(c,k) = r$ and $u \in \Psi_r^-(v)$ as desired.

To this end we show that for all vertices (a,i) on the path in the affine Little graph from (b,j) to (c,k) except (c,k), that $p(a,i) = r$.

Let (a,i) be such a vertex and let $p = p(a,i)$ and $q = q(a,i)$.

For the base case $(a,i) = (b,j)$. Since $b \in R(w)$ we have $p < q$ and since $w \in \Psi_r^+(v)$, $p = r$ as required.

For the induction step suppose (a,i) satisfies $p = r$. Let $(a,i) \to (a',i')$ in the affine Little graph. Write x', y', t', p', q' for the quantities associated with (a',i').

By definition a' is obtained from a by replacing $a_i = t$ by $a_i - 1 = t - 1$. Let $w' = s_{a'_1} \dots s_{a'_l}$. Then $w' = x s_{t-1} y = xy(y^{-1} s_{t-1} y) = v t_{p',q'}$. In particular, since $z s_m z^{-1} = z t_{m,m+1} z^{-1} = t_{z(m),z(m+1)}$ for all $z \in \tilde{S}_n$ and m, we have $\{t-1, t\} = \{y(p'), y(q')\}$ as sets. Let s be such that $y(s) = t-1$. Since $t = y(p) = y(r)$ we have $\{r, s\} = \{p', q'\}$.

Suppose a' is not reduced. By Lemma 21, $i' \neq i$. Suppose $i' < i$. Then $s_{t'} y' < y'$ and $s_{t-1} y > y$. It follows that $q' < p'$ and $s < r$, so that $p' = r$ as desired. Suppose $i' > i$. Then $s_{t'} y' > y'$ and $s_{t-1} y < y$, so that $p' < q'$ and $r < s$ and again $p' = r$ as desired.

Otherwise let a' be reduced. Then $(a',i') = (c,k)$, $i = i'$, and $y' = y$. By the reduced-ness of a', $s_{t-1} y > y$ and $s < r$. Again by reduced-ness $p' < q'$. Therefore $q' = r$ as desired. □

5. Generalized affine Little algorithm

We now generalize the affine Little algorithm of the previous section from reduced words to α-decompositions.

Lemma 12. *Let $v \lessdot w$ both be cyclically decreasing. Then there exists a cyclically decreasing v-marked reduced word a for w which ϕ^v maps to a cyclically decreasing v-marked reduced word, for the element w' say. Furthermore the element w' is independent of the choice of a.*

Proof. Let $w = w(A)$ and $v = w(A - \{i\})$. Let j be maximal such that $\{i, i-1, \dots, i-j\} \subset A$, with indices taken mod n. Let A' be the proper subset of $[0, n-1]$ obtained from A by replacing i with $i-j-1$. Let $w' = w(A')$.

Now take any reduced word a of w and apply ϕ^v to the word w with the reflection i marked. The application of ϕ^v to a replaces the subword $i(i-1)\cdots(i-j)$ by $(i-1)\cdots(i-j-1)$, resulting in a', say, with $i-j-1$ marked. The only way that a' is not cyclically decreasing is if $i-j-2 \in A$ and it appears to the left of $i-j$ in a. In this case $i-j-2$ and $i-j$ are in different maximal cyclically decreasing subintervals of A. By Lemma 1 there is an $a \in R(w)$ with $i-j-2$ to the right of $i-j$. With such a choice of a, by Lemma 1 a' is a reduced word for $w(A')$, which depends only on A', that is, only on A and i. □

In particular, for $v \in \tilde{S}_n$ cyclically decreasing, ϕ^v induces a permutation of the set of cyclically decreasing covers of v. Denote this map by ϕ^v.

Let $v \in \tilde{S}_n$ and $w \in \Psi_r^+(v)$. Let α be fixed. We will describe an algorithm which takes as input an α-decomposition $w = w^1 w^2 \cdots w^r$ of $w \in \Psi_r^+(v)$, and outputs an α-decomposition $x = x^1 \cdots x^r$ of an element $x \in \Psi_r^-(v)$.

Algorithm 13. Initialise $y := w$ and $y^i := w^i$. By Proposition 16 and Corollary 2 we may write $v = y^1 y^2 \cdots (y^i)' \cdots y^r$ where $(y^i)'$ is obtained from y^i by removing a single simple reflection. This simple reflection is the same for any reduced word for y^i.

(1) Treating y^i as a $(y^i)'$-marked word, apply $\phi^{(y^i)'}$ to y^i.
(2) If $y^1 y^2 \cdots \phi^{(y^i)'}(y^i) \cdots y^r$ is an α-decomposition of some permutation (that is, it is "reduced"), we terminate with this as the output. Otherwise by Lemma 21 there is a unique index $j \neq i$ so that

$$v = y^1 y^2 \cdots (y^j)' \cdots \phi^{(y^i)'}(y^i) \cdots y^r$$

where $(y^j)'$ is obtained from y^j by removing a single simple reflection. Replace the α-decomposition $y = y^1 y^2 \cdots y^i \cdots y^r$ by $y^1 y^2 \cdots \phi^{(y^i)'}(y^i) \cdots y^r$ and set $i := j$.
(3) Return to 1.

The fact that the algorithm is well-defined is clear from Lemma 12.

Theorem 14. *Algorithm 13 is a bijection between α-decompositions of permutations in $\Psi_r^+(v)$ and α-decompositions of permutations in $\Psi_r^-(v)$.*

Proof. The fact that the output is an α-decomposition of a permutation in $\Psi_r^-(v)$ follows from the same argument as in Theorem 11, and the fact that ϕ_r is a bijection follows from the reversibility. $\square$

Proof of Theorem 7. Since the coefficient $[x^\alpha]\tilde{F}_w$ of x^α in the affine Stanley symmetric function $\tilde{F}_w$ is given by the number of distinct α-decompositions of w, Theorem 14 proves Theorem 7. $\square$

Remark 15. Given an α-decomposition of w, it is not clear whether there is an initial choice of reduced word a for w so that the affine Little algorithm applied to a naturally gives the same α-decomposition as the generalized affine Little algorithm.

6. A Coxeter-theoretic result

Let (W, S) be a Coxeter system, that is, W is a group generated by a set of *simple reflections* S subject only to relations of the form $s^2 = 1$ for all $s \in S$ and for $s \neq s' \in S$, $(ss')^{m(s,s')} = 1$ for $m(s,s') \geq 2$. W is called a Coxeter group. A *reflection* is by definition a conjugate in W of an element of S.

In this section, to avoid double subscripting, we shall write s_i for an arbitrary simple reflection. A *reduced word* for $w \in W$ is a factorization $w = s_1 \cdots s_r$ of w with $s_i \in S$ such that r is minimum. The number r is the length of w, denoted $\ell(w)$.

Let $v \lessdot w$ denote the covering relation of the *strong Bruhat order* $\leq$ on W. By definition $v \lessdot w$ if and only if there is a reflection $t \in W$ such that $w = vt$ and $\ell(w) = \ell(v) + 1$.

The goal of this section is to establish Lemma 21, which is used crucially in the affine Little bijection.

Proposition 16. *(Strong Exchange Condition)* [4, Theorem 5.8] *Let $w = s_1 s_2 \cdots s_r$ with $s_i \in S$ not necessarily reduced. Suppose t is a reflection such that $\ell(wt) < \ell(w)$. Then there is an index i such that $wt = s_1 \cdots \hat{s}_i \cdots s_r$ where the hatted reflection is omitted. Moreover if the expression for w is reduced then the index i is unique.*

Corollary 17. [4, Cor. 5.8] *Suppose $w = s_1 \cdots s_r$ with $s_i \in S$ and $r > \ell(w)$. Then there are indices $i < j$ for which $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r$.*

Lemma 18. [4, Lemma 5.11] *Let $w' \lessdot w$. Suppose there is an $s \in S$ such that $w' < w's$ and $w's \neq w$. Then both $w < ws$ and $w's < ws$.*

Lemma 19. *Let $w = s_1 \cdots s_r$ with $s_i \in S$ and $1 \leq i \leq r$. Then there are unique reflections t, t' such that $wt = t'w = s_1 \cdots \hat{s}_i \cdots s_r$.*

Proof. Let $x = s_1 \cdots s_{i-1}$ and $y = s_{i+1} \cdots s_r$. Then $s_1 \cdots \hat{s}_i \cdots s_r = xy$ and $w = xsy = xy(y^{-1}sy)$. Letting $t = y^{-1}sy$ and $t' = xsx^{-1}$ we have $wt = xy$ and $t'w = xsx^{-1}xsy = xy$ as desired. □

Lemma 20. *Let $x, y \in W$. Then $\ell(xy) < \ell(x) + \ell(y)$ if and only if there is a reflection t such that $xt < x$ and $ty < y$.*

Proof. For the converse, if t is a reflection such that $xt < x$ and $ty < y$, then $\ell(xy) = \ell(xtty) \leq \ell(xt) + \ell(ty) \leq \ell(x) - 1 + \ell(y) - 1$ as desired.

Suppose $x, y \in W$ are such that $\ell(xy) < \ell(x) + \ell(y)$. Let $x = s_1 \cdots s_l$ and $y = s'_1 \cdots s'_m$ be reduced. Then $xy = s_1 \cdots s_l s'_1 \cdots s'_m$ is not reduced. By Corollary 17, xy has a factorization obtained by removing two of the simple reflections. Suppose both of the reflections are removed from the reduced word for x, that is, $xy = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_l s'_1 \cdots s'_m$ for some $1 \leq i < j \leq l$. Multiplying both sides by $y^{-1} = s'_m \cdots s'_1$ we have $x = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_l$ which contradicts the assumption that $x = s_1 \cdots s_l$ was reduced. Similarly both reflections cannot be removed from the reduced word for y. Therefore there are indices $1 \leq i \leq l$ and $1 \leq j \leq m$ such that $xy = s_1 \cdots \hat{s}_i \cdots s_l s'_1 \cdots \hat{s'_j} \cdots s'_m$. By Lemma 19 there are reflections $t, t' \in W$ such that $xt = s_1 \cdots \hat{s}_i \cdots s_l$ and $t'y = s'_1 \cdots \hat{s'_j} \cdots s'_m$. Therefore $xtt'y = s_1 \cdots \hat{s}_i \cdots s_l s'_1 \cdots \hat{s'_j} \cdots s'_m = xy$. It follows that $tt' = 1$ and $t = t'$ since t, t' are reflections. Since xt admits a shorter factorization in simple reflections that x does, $xt < x$. Similarly $ty = t'y < y$ as desired. □

Lemma 21. *Suppose $s_i \in S$ for $1 \leq i \leq r$ is such that $v = s_1 \cdots \hat{s}_i \cdots s_r$ is reduced, where $\hat{s}_i$ means that the factor s_i is removed. Suppose also that $w = s_1 \cdots s_r$ is not reduced. Then there is a unique index $j \neq i$ such that $s_1 \cdots \hat{s}_j \cdots s_r$ is reduced. Moreover $v = s_1 \cdots \hat{s}_j \cdots s_r$.*

Proof. Let $x = s_1 \cdots s_{i-1}$ and $y = s_{i+1} \cdots s_r$. Since $v = xy = s_1 \cdots \hat{s}_i \cdots s_r$ is reduced, both of the expressions for x and y are reduced and $\ell(xy) = \ell(x) + \ell(y)$. By Lemma 20 for $x, y \in W$ and the reflection s_i, either $xs_i > x$ or $s_i y > y$. Without loss of generality we assume that $xs_i > x$; one may reduce to this case by reversing the reduced words and replacing elements of W by their inverses.

By assumption $xs_i = s_1 \cdots s_i$ is reduced. If $s_1 \cdots s_r$ is reduced then we are done. Suppose not. Let j be the minimum index such that $i+1 \leq j \leq l$ and $s_1 \cdots s_j$ is not reduced. Then $s_1 \cdots s_{j-1} \gtrdot s_1 \cdots \hat{s}_i \cdots s_{j-1}$. Applying Lemma 18 to this covering relation and the simple reflection s_j, we have either (a) $s_1 \cdots \hat{s}_i \cdots s_j = s_1 \cdots s_{j-1}$ or (b) $s_1 \cdots s_j \gtrdot s_1 \cdots \hat{s}_i \cdots s_j$. If (b) holds, since $s_1 \cdots \hat{s}_i \cdots s_j$ is reduced it follows that $s_1 \cdots s_j$ is also, contradicting our choice of j. So (a) must hold. Right multiplying by $s_{j+1} \cdots s_r$ we have $v = s_1 \cdots \hat{s}_i \cdots s_r = s_1 \cdots \hat{s}_j \cdots s_r$. But $s_1 \cdots \hat{s}_i \cdots s_r$ is reduced and $s_1 \cdots \hat{s}_j \cdots s_l$ has the same number of reflections, so it too must be reduced.

For uniqueness, suppose there is a k distinct from i and j such that $v = s_1 \cdots \hat{s_k} \cdots s_r$ is reduced. We now treat the three indices i, j, k interchangeably and suppose without loss of generality that $i < j < k$. Using the reduced words for v that omit s_i and s_j, we have $\hat{s}_i \cdots s_j = s_i \cdots \hat{s}_j$. Right multiplying by s_j we have $\hat{s}_i \cdots \hat{s}_j = s_i \cdots s_j$. But $s_i \cdots s_j$ is reduced, being a subword of the reduced word $s_1 \cdots \hat{s_k} \cdots s_r$. This is a contradiction. □

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